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On the Diffusion of a Fast Molecule

S. Goldstein[1,*] and J. Guetti[2,†]

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We consider the motion of a heavy particle in interaction with an infinite ideal gas of slow atoms. We prove that the velocity of the heavy particle is, in a suitable limit, modeled by a deterministic process. We also treat the process of rescaled velocity fluctuations around a certain deterministic motion and show that this is appropriately modeled by a nonhomogeneous diffusion process.

KEY WORDS: Nonhomogeneous diffusion; velocity fluctuations; Rayleigh piston; Markov approximation; deterministic limit; fluctuations.

1. INTRODUCTION

Attention has recently been given to modeling the motion of a heavy particle (piston, molecule) in a gas of atoms by a diffusion process. Using an uncontrolled Markovian approximation, Miller and Stein argue in Ref. 6 that the behavior of the piston can be divided into three regimes. First, if the speed of the piston is much less than the speed of the atoms, its velocity should be modeled by the Ornstein–Uhlenbeck process. When the piston's speed is on the order of or much larger than the speed of the atoms, the appropriate model for the velocity of the piston is deterministic. In these latter cases, they also argue that the rescaled velocity fluctuations around the deterministic motion should be modeled by nonhomogeneous diffusion processes.

The motion of the slow molecule has been rigorously (i.e., with no Markovian assumption) treated by Holley[4] in one dimension and by

[1] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903.

[2] Department of Mathematics, Seton Hall University, South Orange, New Jersey 07079.

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Dürr, Goldstein, and Lebowitz[2] in any dimension. They establish the convergence of the mechanical, non-Markovian process describing the velocity of the molecule to the Ornstein–Uhlenbeck process in a limit in which the molecule becomes very heavy.

We here treat rigorously the motion of a fast molecule in one dimension. The physical situation is as follows: We consider the motion of a heavy molecule of mass M in an infinite ideal gas of point particles of mass m with which the molecule interacts via elastic collisions. The initial position and velocity of the molecule may be chosen arbitrarily. We wish to describe the motion of the molecule in the limit where $m \to 0$, the gas has density $\rho \sim m^{-1}$, and velocity distribution $f_m(v) = m^{-\gamma} f(m^{-\gamma} v)$ for $\gamma > 0$, where $f(v)$ is absolutely continuous with respect to Lebesgue measure and has all moments finite. We thus consider the sequence of stochastic processes $V_{m,t}$ where $V_{m,t}$ is the velocity of the molecule at time t in a bath of atoms of mass m. We prove in the limit $m \to 0$ that $V_{m,t} \to \bar{V}(t)$, the limiting deterministic process. We also consider the scaled deviations $\xi_m(t)$ of $V_{m,t}$ from a suitable deterministic process $\bar{V}_{m,t}$, depending on m, which converges to $\bar{V}(t)$ as $m \to 0$. We show $\xi_m(t)$ converges in distribution to a nonhomogeneous diffusion process.

We also wish to note that for $\gamma > \frac{1}{2}$, the scaled deviations could have been defined with respect to $\bar{V}(t)$, à la Van Kampen[7]; and to emphasize that for $\gamma < \frac{1}{2}$, the scaled deviations *from* $\bar{V}(t)$ are divergent as $m \to 0$ since the fluctuations of $V_{m,t}$ around its mean $\langle V_{m,t} \rangle$, which are of order $\sqrt{m}$, are much smaller than the distance of $\langle V_{m,t} \rangle$ (equivalently of $\bar{V}_m(t)$) from $\bar{V}(t)$, which is of order m^{γ} in this case.

The outline of the paper, which is based on the techniques of Ref. 2, is as follows: In Section 2 we describe the model more precisely. In Section 3 the main results are presented. Sections 4–6 contain the proofs of the results. In Section 6 we show that the mechanical model can be well approximated by a Markov process as $m \to 0$, the central idea in the proof.

Though we have not examined carefully the case of two or more dimensions, we believe our results should extend to these cases.

2. THE MECHANICAL MODEL

In what follows the heavy point particle will be called the molecule and the light particles will be called atoms. Let $\Gamma = \mathbb{R} \times \mathbb{R}$ denote the one particle phase space, $\mathscr{B}(\Gamma)$ its Borel algebra, and μ_m the absolutely continuous measure on Γ defined by

$$d\mu_m = \rho_m \, dq \, f_m(v) \, dv \qquad q, v \in \mathbb{R} \tag{2.1}$$

where

$$\rho_m = \frac{\rho}{\alpha}, \qquad \rho > 0 \qquad \text{and} \qquad \alpha = \frac{m}{m+M} \tag{2.2}$$

and

$$f_m(v) = m^{-\gamma} f(m^{-\gamma} v) \qquad \gamma > 0 \tag{2.3}$$

We assume that the velocity distribution has all moments finite, i.e., $\int |v|^n f(v)\, dv < \infty$ for $n \in \mathbb{Z}^+$, as, for example, in the Maxwellian distribution where $f(v) = (m\beta/2\pi)^{1/2} e^{-\beta m v^2/2}$, $\beta > 0$.

An ideal gas of atoms of mass m is described by a Poisson field $(\Omega, \mathscr{F}, \mathbb{P}_m)$ built on $(\Gamma, \mathscr{B}(\Gamma), \mu_m)$: If $N_B =$ the number of atoms with coordinates $(q, v) \in B$, $B \in \mathscr{B}(\Gamma)$, then N_B is Poisson with mean $\mu_m(B)$

$$\mathbb{P}_m\{[\omega \in \Omega \mid N_B(\omega) = k]\} = \exp[-\mu_m(B)] \frac{\mu_m^k(B)}{k!} \tag{2.4}$$

where ω represents a configuration of countably many atoms, i.e., $\omega = (q_i, v_i)_{i \in \mathbb{N}}$. It follows that if $B_1, \ldots, B_n$ are pairwise disjoint sets that the random variables N_{B_i} are independent. We can think of this Poisson field as describing atoms independently distributed in position space with density ρ_m and having independent velocities distributed according to $f_m(v)$.

The scaling of $f(v)$ is such that the speed of the atoms is like m^γ and hence tends to zero as m tends to zero.

Let us put a molecule of mass M, $M > m$, at position X^0 with velocity V^0, $V^0 > 0$, in the gas. Assuming the collisions between the molecule and the atoms are elastic, conservation of energy and momentum relate the precollision (V, v) and post-collision (V', v') velocities as follows:

$$\begin{aligned} V' &= V + 2\alpha(v - V) \\ v' &= v + 2\alpha(V - v)(M/m) \end{aligned} \tag{2.5}$$

For each $\omega \in \Omega$, we can define the velocity of the molecule $V_{m,t}(\omega)$ $[\equiv V_m(t, \omega)]$ as a right continuous function of t in this way: The molecule starts with velocity V^0. Let $\tau_1(\omega)$ denote the time of the first collision. $V_{m,t}(\omega) = V^0$ for all $t < \tau_1(\omega)$ and then changes to a new velocity V' according to (2.5). Afterwards the molecule moves freely with velocity V' until $\tau_2(\omega)$, the time of the next collision.

Infinitely many collisions in a finite amount of time, as well as simultaneous collisions of two or more atoms with the molecule can be ignored,[2,8] so that the motion of the molecule is well defined by the above prescription.

For any $I=[0, T]$, $0<T<\infty$, let $D(I)$ denote the space of right continuous functions with left limits defined on I, equipped with the Skorohod topology.[1] Let $\mathscr{B}[D(I)]$ denote the Borel σ algebra.

We can define $V_{m,t}$ as a stochastic process $V_m: \Omega \to D(I)$, where $V_m(\omega)=V_{m,\cdot}(\omega)$. V_m induces a measure P_m on $D(I)$:

$$P_m(A)=\mathbb{P}_m(\{\omega \mid V_m(\omega)\in A\}) \qquad \text{for all} \quad A\in \mathscr{B}[D(I)]$$

Note that V_m is not Markovian since there are configurations of the bath which lead to recollisions between the atoms and the molecule.

We are now ready to state our results.

3. RESULTS

Let $\bar{V}$ be the deterministic process described by the differential equation

$$\frac{d\bar{V}}{dt}=-2\rho\bar{V}\,|\bar{V}| \qquad \text{where} \qquad \bar{V}(0)=V^0 \tag{3.1}$$

with solution

$$\bar{V}(t)=\frac{V^0}{1+2\rho\,|V^0|\,t} \tag{3.2}$$

Let P_m $(m\to 0)$ and P be probability measures on $\{D(I), \mathscr{B}[D(I)]\}$. P_m is said to converge weakly to P $(P_m \Rightarrow_{m\to 0} P)$ if for all bounded, continuous real functions h on $D(I)$, $\lim_{m\to 0}\int h\,dP_m=\int h\,dP$. Equivalently, $\liminf_m P_m(G)\geqslant P(G)$ for all open sets G. We say that the process $X_m(t)$, $t\geqslant 0$, converges in distribution to the process $X(t)$, $t\geqslant 0$, and write $X_m \Rightarrow^{\mathscr{D}}_{m\to 0} X$, if for each interval, I, $P^{X_m}\Rightarrow_{m\to 0} P^X$, where $P^{X_m}(P^X)$ is the distribution of $X_m(X)$ on $D(I)$.

Theorem 3.1. $V_m \Rightarrow^{\mathscr{D}}_{m\to 0} \bar{V}$, where V_m is the molecular velocity process with $V_m(0)=V^0$ and $\bar{V}$ is the deterministic process described by (3.1).

Let $\bar{V}_m(t)$ be the deterministic process satisfying the differential equation

$$\frac{d\bar{V}_m}{dt}=\phi_m(\bar{V}_m) \tag{3.3}$$

where

$$\phi_m(\bar{V}_m)=2\rho\int_{-\infty}^{\infty}|v-\bar{V}_m|\,(v-\bar{V}_m)\,f_m(v)\,dv \tag{3.4}$$

and define the fluctuation process ξ_m by

$$\xi_m(t) = \frac{V_m(t) - \bar{V}_m(t)}{\sqrt{\alpha}} \tag{3.5}$$

The ξ_m process describes the scaled deviations of the molecular velocity from $\bar{V}_m$, which may be interpreted as the "deterministic part" of V_m (see Eq. (5.4)). It follows easily, e.g., from the proof of Theorem 3.1, that $\bar{V}_m$ approaches $\bar{V}$ as $m \to 0$.

Let $\xi(t)$ be the nonhomogeneous diffusion process whose generator B restricted to the set of infinitely differentiable functions of compact support, is given by

$$B_t \mid_{C_c^\infty} = -4\rho\xi \bar{V}(t) \frac{\partial}{\partial\xi} + 2\rho \, |\bar{V}(t)|^3 \frac{\partial^2}{\partial\xi^2} \tag{3.6}$$

Theorem 3.2. $\xi_m(t) \Rightarrow^{\mathscr{D}}_{m \to 0} \xi(t)$ where $\xi_m(0) = \xi(0) = 0$.

As in Ref. 2 we are concerned with the convergence of non-Markovian processes to a Markov process. Employing the same techniques as in Ref. 2 we will prove our theorems in two steps. To prove Theorem 3.1 we will first modify the mechanical process and consider an abstract Markov process $\tilde{V}_m$ for which we will prove

$$\tilde{V}_m \Rightarrow^{\mathscr{D}}_{m \to 0} \bar{V} \tag{3.7}$$

The second step will be to establish the closeness of paths of a suitable realization V'_m (possibly depending on I) of $\tilde{V}_m$ to the paths of V_m as $m \to 0$. V'_m and V_m are realized on the same probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}_m)$ in such a way that for all $\varepsilon > 0$ and any I

$$\lim_{m \to 0} \bar{\mathbb{P}}_m(\{\bar{\omega} \in \bar{\Omega} \mid \sup_{t \in I} |V'_m(t, \bar{\omega}) - V_m(t, \bar{\omega})| \geqslant \varepsilon\}) = 0 \tag{3.8}$$

Theorem 3.1 follows easily from (3.7) and (3.8). Theorem 3.2, though slightly more complicated since $\xi(t)$ is time inhomogeneous, is proved by similar considerations.

4. THE MARKOV APPROXIMATION

We shall now define the Markov process $\tilde{V}_m$.

Lemma 4.1. Suppose that at time t the molecule has velocity V and is surrounded by a bath of atoms having a Poisson distribution described in (2.4). The probability $p_m(dt, dv, V)$ for a collision between an atom with velocity $v \in dv$ and the molecule in time $(t, t+dt)$ is given by

$$p_m(dt, dv, V) = \rho_m \, |v - V| \, dt \, f_m(v) \, dv \tag{4.1}$$

Proof. Recall that the probability of finding a particle in the $q-v$ phase space is given by $\rho_m \, dq \, f_m(v) \, dv$. For the occurrence of a collision in the $v-t$ phase space we get $\rho_m \, |v-V| \, dt \, f_m(v) \, dv$.

We define an abstract Markov process using the collision rates of (4.1). More precisely, we set

$$N_m(V) = \int_{-\infty}^{\infty} |v - V| \, f_m(v) \, dv \tag{4.2}$$

and let

$$g_m(V, v) = \frac{1}{N_m(V)} |v - V| \, f_m(v) \tag{4.3}$$

Set

$$\bar{\rho}_m(V) = \rho_m N_m(V) \tag{4.4}$$

Note that $\bar{\rho}_m(V) \, dt \, g_m(V, v) \, dv = \rho_m(dt, dv, V)$. Using the collision equations (2.5) we can transform $g_m(V, v)$ into a transition kernel $\mathscr{G}_m(V, dV)$ giving the jump probability from V to $V' \in dV$ in one collision.

Definition 4.1. Let $\tilde{V}_m$ be the Poisson jump process defined on the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}_m)$ with mean waiting time $\bar{\rho}_m(V)^{-1}$ and transition probability $\mathscr{G}_m(V, dV)$ and $\tilde{V}_m(0) = V^0$. Let $\tilde{\mathbb{P}}_m$ denote the measure induced by $\tilde{V}_m$ on $\{D(I), \mathscr{B}[D(I)]\}$. Let $\tilde{\xi}_m = (\tilde{V}_m - \bar{V}_m)/\sqrt{\alpha}$ be the fluctuation process for the Markov approximation.

In Section 5 we will prove the following two lemmas:

Lemma 4.2. $\tilde{V}_m \Rightarrow^{\mathscr{D}}_{m \to 0} \bar{V}$.

Lemma 4.3. $\tilde{\xi}_m \Rightarrow^{\mathscr{D}}_{m \to 0} \xi$.

5. PROOFS OF LEMMAS 4.2 AND 4.3

Let us first recall some basic facts about probability semigroups. What is presented here may be found in Dynkin[3] and Kurtz.[5] Let $\underline{Z}$ denote a Markov process on $D(I)$ having transition probability $Q_t(x, dy)$. To $\underline{Z}$

corresponds a contraction semigroup T_t defined on B, the Banach space of bounded, measurable functions h: $\mathbb{R} \to \mathbb{R}$ with sup norm $\|\cdot\|$,

$$T_t h(x) = \int_{\mathbb{R}} h(y)\, Q_t(x, dy)$$

Let C_0 denote the Banach subspace of B consisting of continuous functions vanishing at infinity. Suppose C_0 is invariant under T_t and T_t on C_0 is continuous: $T_t C_0 \subseteq C_0$ and $\lim_{t\to 0} \|T_t h - h\| = 0$ for all $h \in C_0$. Then $\underline{Z}$ is a Markov C_0 process.

We define the (infinitesimal) generator A of the semigroup T_t by

$$Ah = \lim_{t \to 0} \frac{T_t h - h}{t} \tag{5.1}$$

and $\mathscr{D}_A$, the domain of A, consists of all $h \in B$ (C_0 for a Markov C_0 process) for which the limit exists in the sup-norm topology.

In order to prove Lemmas 4.2 and 4.3 we will use a theorem due to Kurtz[5] formulated in a more general setting which relates convergence of processes in distribution to strong convergence of generators on a core.

Lemma 5.1. Consider a sequence $\underline{Z}_n$ of Markov processes with sample paths in $D(I)$ and generators A_n. Suppose $\underline{Z}$ is a Markov C_0 process with sample paths in $D(I)$ and generator A. Let K be a core for A and suppose that $h \in K$ implies that $h \in \mathscr{D}(A_n)$ for all sufficiently large n. Suppose that the initial distributions of $\underline{Z}_n$ converge weakly to the initial distribution of $\underline{Z}$ and that

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}} |A_n h(x) - Ah(x)| = 0 \tag{5.2}$$

for all $h \in K$. Then $\underline{Z}_n \Rightarrow^{\mathscr{D}}_{n\to\infty} \underline{Z}$.

Remark. K is a core for A if K is subspace of $\mathscr{D}_A$ such that A is the closure of $A\,|_K$. If K is a linear subspace contained in $\mathscr{D}_A$ such that K is dense and T_t: $K \to K$, then K is a core for A.

For the processes $\bar{V}$ and ξ, $T_t C_0^2 \subseteq C_0^2$, where C_0^2 is the set of twice continuously differentiable functions vanishing at infinity. Moreover, C_0^2 is dense in C_0 and is contained in $\mathscr{D}_A$ so that C_0^2 is a core for the generators of $\bar{V}$ and ξ. It then easily follows that C_c^∞, the infinitely differentiable functions of compact support, is also a core.

We now prove Lemma 4.2. Let $\tilde{A}_m$ be the generator and $\tilde{T}_t^{(m)}$ the semigroup of $\tilde{V}_m$. The generator A of $\bar{V}$ is given by $A\,|_{C_c^\infty} = -2\rho\,|\bar{V}|\,\bar{V}\,\partial/\partial\bar{V}$. The initial distribution for both $\tilde{V}_m$ and $\bar{V}$ is given by $\delta(V-V^0)$. Thus by Lemma 5.1 it suffices to show that $\lim_{m\to 0} \|\tilde{A}_m h - Ah\| = 0$ for $h \in C_c^\infty$.

We first compute $\tilde{A}_m$. Let us denote by $\tilde{E}^{(m)}_{V^0}(\cdot) = \tilde{E}^{(m)}(\cdot \mid \tilde{V}_m(0) = V^0)$, the expectation for $\tilde{V}_m$ starting at V^0. Then $\tilde{T}^{(m)}_t h(V^0) = \tilde{E}^{(m)}_{V^0}(h(\tilde{V}_{m,t}))$ for $h \in B$. Thus

$$\tilde{A}_m h(V^0) = \lim_{t\to 0} \frac{1}{t} [\tilde{E}^{(m)}_{V^0} h(\tilde{V}_{m,t}) - h(V^0)]$$

for $h \in \mathscr{D}_{\tilde{A}_m}$.

For a Poisson process the probability of more than one jump in time t is of order $O(t^2)$. Hence in the expectation we need only consider terms where no more than one jump has taken place in time t. Using Definition 4.1 we obtain

$$\tilde{A}_m h(V^0) = -\bar{\rho}_m(V^0)\, h(V^0) + \bar{\rho}_m(V^0) \int \mathscr{G}_m(V^0, dV')\, h(V') \tag{5.3}$$

We now consider the limit $m \to 0$. Recall that $\rho_m = \rho/\alpha$, $\alpha = m/(m+M)$, and $V' = V^0 + 2\alpha(v - V^0)$. The Taylor expansion for h around V^0 gives

$$h(V') = h(V^0) + h'(V^0)[2\alpha(v-V^0)] + h''(V^0)[2\alpha^2(v-V^0)^2] + h'''(\hat{V})[\tfrac{4}{3}\alpha^3(v-V^0)^3]$$

where $\hat{V} = V^0 + \delta[2\alpha(v-V^0)]$, $\delta \in [0, 1]$. Placing this into (5.3) and rewriting $\mathscr{G}_m(V^0, dV')$ as $g_m(V^0, v)\, dv$, we obtain

$$\begin{aligned}\tilde{A}_m h(V^0) &= 2\rho \int_{-\infty}^{\infty} |v-V^0|\, (v-V^0)\, h'(V^0)\, f_m(v)\, dv \\ &\quad + 2\rho\alpha \int_{-\infty}^{\infty} |v-V^0|\, (v-V^0)^2\, h''(V^0)\, f_m(v)\, dv \\ &\quad + \frac{4}{3}\rho\alpha^2 \int_{-\infty}^{\infty} |v-V^0|\, (v-V^0)^3\, h'''(\hat{V})\, f_m(v)\, dv \end{aligned} \tag{5.4}$$

In the computations that follow we will use the fact that

$$\int |v-V^0|\, (v-V^0)^k f_m(v)\, dv = \int |m^\gamma u - V^0|\, (m^\gamma u - V^0)^k f(u)\, du \tag{5.5}$$

and we will denote

$$\int_{-\infty}^{\infty} \cdot f_m(v)\, dv \text{ by } \langle\cdot\rangle_m \qquad \text{and} \qquad \int_{-\infty}^{\infty} \cdot f(u)\, du \text{ by } \langle\cdot\rangle \tag{5.6}$$

Since $h \in C_c^\infty$, we may assume $h(x)=0$ for $|x| \geqslant b$. Also $\sup_{V^0 \in \mathbb{R}} |h''(V^0)| < \infty$. First we consider the term $2\rho\alpha \int_{-\infty}^{\infty} |v - V^0| (v - V^0)^2 h''(V^0) f_m(v)\, dv$. Since the integral vanishes for $|V^0| > b$, it tends to zero uniformly in V^0 as $\alpha \to 0$.

Next we consider $\frac{4}{3}\rho\alpha^2 \int_{-\infty}^{\infty} |v - V^0| (v - V^0)^3 h'''(\hat{V}) f_m(v)\, dv$. For $|V^0| < b+1$, this integral converges to zero uniformly in V^0. Suppose $|V^0| > b+1$ so that V^0 is "firmly" in the complement of the support of h. Since $\hat{V} = V^0 + 2\alpha\delta(v - V^0)$, v must be such that for the post-collision velocity V we have $|V| < b$. We may assume that $V^0 > b+1$, the case $V^0 < -(b+1)$ being similar. Then, for such a v, we have using (2.5)

$$v \leqslant \frac{b}{2\alpha} - V^0 \left(\frac{1}{2\alpha} - 1 \right) < -V^0$$

for α small enough. Therefore, for $V^0 > b+1$

$$\begin{aligned}
\frac{4}{3}\rho\alpha^2 & \left| \int_{-\infty}^{-V^0} |v - V^0| (v - V^0)^3 h'''(\hat{V}) f_m(v)\, dv \right| \\
&= \frac{4}{3}\rho\alpha^2 \int_{-\infty}^{-V^0} |v - V^0|^4 |h'''(\hat{V})| f_m(v)\, dv \\
&\leqslant \frac{4}{3}\rho\alpha^2 \sup_{\hat{V} \in \mathbb{R}} |h'''(\hat{V})| \int_{-\infty}^{-V^0} (2|v|)^4 f_m(v)\, dv \\
&\leqslant \frac{64}{3}\rho\alpha^2 \sup_{\hat{V} \in \mathbb{R}} |h'''(V)|\, m^{4\gamma} \langle u^4 \rangle
\end{aligned}$$

which tends to zero as $\alpha \to 0$.

The remaining term is $2\rho \int_{-\infty}^{\infty} |v - V^0| (v - V^0) h'(V^0) f_m(v)\, dv$. Using (2.3) and the fact that f has finite moments, we find that this integral tends to $-2\rho |V^0| V^0 h'(V^0)$ as $m \to 0$ uniformly in V^0. Thus

$$\lim_{m \to 0} \sup_{V \in \mathbb{R}} |\tilde{A}_m h(V) - Ah(V)| = 0 \qquad \text{for all} \quad h \in C_c^\infty$$

We now turn to the proof of Lemma 4.3. We would like to simply repeat the previous steps and apply Lemma 5.1. However, some modifications are necessary as $\xi(t)$ is a nonhomogeneous diffusion process, as can be seen from its generator, B_t. We can treat this situation by first passing to an extended state space. That is, we consider the process

$$Y_m(t) = (\theta(t), \tilde{\tilde{\xi}}_m(t)) \tag{5.7}$$

where $\theta(t)=\theta+t$, $\theta\geqslant 0$, and $\tilde{\tilde{\xi}}_m(t)=[\tilde{V}_m(t)-\bar{V}_m(\theta(t))]/\sqrt{\alpha}$, with $\bar{V}_m(0)=V^0$ fixed, so that different initial values $\tilde{\tilde{\xi}}_m(0)$ correspond to different values of $\tilde{V}_m(0)$. Let $y=(\theta(0), \tilde{\tilde{\xi}}_m(0))=(\theta, \xi)$, and let $V^*=\tilde{V}_m(0)$ so that $V^*=\sqrt{\alpha}\,\xi+\bar{V}_m(0)$.

We show that $Y_m(t)\Rightarrow^{\mathscr{D}}_{m\to 0} Y(t)\equiv(\theta(t), \xi(t))$, which has the generator

$$B=\frac{\partial}{\partial\theta}-4\rho\xi\,|\bar{V}(\theta)|\,\frac{\partial}{\partial\xi}+2\rho\,|\bar{V}(\theta)|^3\,\frac{\partial^2}{\partial\xi^2} \tag{5.8}$$

Let $h\colon \mathbb{R}^2\to\mathbb{R}$ such that $h\in C_c^\infty$. Then for the generator $\tilde{B}_m$ of Y_m we have

$$\tilde{B}_m h(y)=\lim_{t\to 0}\frac{1}{t}\,[\tilde{E}_y^{(m)}h(Y_m(t))-h(y)]$$

for $h\in\mathscr{D}_{\tilde{B}_m}$, where $\tilde{E}_y^{(m)}$ is the expectation for Y_m starting from y. In the expectation we consider two types of changes in ξ_m, those due solely to changes in $\bar{V}_m$ and those due to jumps in $\tilde{V}_m$. As before we need only consider those terms where no more than one jump has taken place in time t.

Using Definition 4.1, we find

$$\begin{aligned}\tilde{B}_m h(y)=&\frac{\partial h}{\partial\theta}(y)-\frac{2\rho}{\sqrt{\alpha}}\int_{-\infty}^{\infty}|v-\bar{V}_m(\theta)|\,[v-\bar{V}_m(\theta)]\,f_m(v)\,dv\,\frac{\partial h}{\partial\xi}(y)\\&-\bar{\rho}_m(V^*)\,h(y)+\rho_m\int_{-\infty}^{\infty}|v-[\sqrt{\alpha}\,\xi+\bar{V}_m(\theta)]|\,f_m(v)\\&\times h\{\theta, \xi+2\sqrt{\alpha}\,[v-(\sqrt{\alpha}\,\xi+\bar{V}_m(\theta))]\}\,dv\end{aligned}$$

Again expanding $h\{\theta, \xi+2\sqrt{\alpha}\,[v-(\sqrt{\alpha}\,\xi+\bar{V}_m(\theta))]\}$ in a Taylor series about ξ we obtain

$$\begin{aligned}\tilde{B}_m h(y)=&\frac{\partial h}{\partial\theta}(y)-\frac{2\rho}{\sqrt{\alpha}}\int_{-\infty}^{\infty}|v-\bar{V}_m(\theta)|\,[v-\bar{V}_m(\theta)]\,f_m(v)\,dv\,\frac{\partial h}{\partial\xi}(y)\\&-\frac{\rho}{\alpha}\int_{-\infty}^{\infty}|v-V^*|\,f_m(v)\,dv\,h(y)+\frac{\rho}{\alpha}\int_{-\infty}^{\infty}|v-\bar{V}_m(\theta)-\sqrt{\alpha}\,\xi|\,f_m(v)\\&\times\Big\{h(y)+2\sqrt{\alpha}\,[v-\sqrt{\alpha}\,\xi-\bar{V}_m(\theta)]\,\frac{\partial h}{\partial\xi}(y)+2\alpha[v-\sqrt{\alpha}\,\xi-\bar{V}_m(\theta)]^2\\&\times\frac{\partial^2 h}{\partial\xi^2}(y)+\frac{4}{3}\,\alpha^{3/2}[v-\sqrt{\alpha}\,\xi-\bar{V}_m(\theta)]^3\,\frac{\partial^3 h}{\partial\xi^3}(\hat{y})\Big\}\,dv\end{aligned}$$

where $\hat{y}=y+2\delta\alpha[v-\sqrt{\alpha}\,\xi+\bar{V}(\theta)]$, $\delta\in[0, 1]$ (5.9)

By (2.3) and arguments for uniform convergence similar to those used in the proof of Lemma 4.2 we find that (5.9) converges as $m \to 0$ to

$$\frac{\partial h}{\partial \theta}(y) - 4\rho\xi\, |\bar{V}(\theta)|\, \frac{\partial h}{\partial \xi}(y) + 2\rho\, |\bar{V}(\theta)|^3 \frac{\partial^2 h}{\partial \xi^2}(y) \tag{5.10}$$

uniformly in y. Thus $\lim_{m \to 0} \sup_{y \in \mathbb{R}^2} |\tilde{B}_m h(y) - Bh(y)| = 0$ and $Y_m \Rightarrow^{\mathscr{D}}_{m \to 0} Y$ (provided they both start from the same point). By projecting on the ξ component of $Y(t)$ and considering the initial condition $(\theta, \xi) = (0, 0)$, we obtain that $\tilde{\xi}_m(t) \Rightarrow^{\mathscr{D}}_{m \to 0} \xi(t)$, the nonhomogeneous diffusion whose generator was given in (3.6).

6. CLOSENESS OF PATHS

For convenience we assume throughout this section that $V^0 \geqslant 0$, and fix an interval $I = [0, T]$. The goal is now to define a realization $V'_m(\xi'_m)$ of the Markov process $\tilde{V}_m(\tilde{\xi}_m)$ which is close to the mechanical process $V_m(\xi_m)$ in the sense of (3.8). Such a simultaneous realization or coupling of the processes $\tilde{V}_m$ and V_m ($\tilde{\xi}_m$ and ξ_m) we call a *good* coupling.

Recall that the Markov process $\tilde{V}_m$ is a Poisson jump process with mean waiting time $\bar{\rho}_m(V)^{-1}$ and jump distribution determined by $g_m(V, v)\, dv$, i.e., it may be characterized by the (collision) rates

$$r_m(V, v) = \bar{\rho}_m(V)\, g_m(V, v) \tag{6.1}$$

We obtain a good coupling by constructing V'_m in such a way that most collisions for V'_m—certain "good" collisions—are collisions for V_m. On I, until $V_m(t) \leqslant \bar{V}(T)/4$ any atom with velocity $v < \bar{V}(T)/4$ which collides with the molecule cannot have collided earlier. Hence until $V_m(t) \leqslant \bar{V}(T)/4$ collisions between the molecule and atoms with velocity $v < \bar{V}(T)/4$ are governed by the rates (6.1). Note that for $v > \bar{V}(T)/4$ prior collisions may be possible, so that (6.1) does not describe the mechanical process even before $V_m(t) \leqslant \bar{V}(T)/4$.

In the following we denote by (Me) the mechanical molecule, in the mechanical process V_m, and by (Ma) the Markov molecule, undergoing the Markov process V'_m, which we now define. We use $V(V')$ as the generic variable for the velocity of $(Me)((Ma))$. (Me) and (Ma) have the same initial conditions.

Definition 6.1. A collision between an atom and the molecule in which $v < \bar{V}(T)/4$ is called a *good* collision. A collision in which $v \geqslant \bar{V}(T)/4$ is called a *bad* collision.

Until (Me) has velocity $V \leqslant \bar{V}(T)/4$, good collisions do not occur as recollisions (see Remark 6.2). But bad collisions may occur as recollisions, either real or "virtual." The latter are those which are impossible given the past history of the molecule.

Let $\tau = \inf\{t \geqslant 0 \mid V_m(t) \leqslant \bar{V}(T)/4\}$. τ plays the role of a "decoupling time" in the following prescription for V'_m.

1. Given a configuration ω, we observe the velocity of (Me), and when (Me) has a good collision, the velocity of (Ma) is changed according to (2.5) as if it too had the same collision, i.e., collided with an atom with the same velocity.
2. Bad collisions for (Ma) and (Me) occur independently with the rate for bad collisions for V'_m given by (6.1), the collision rate for the Markov process, $\tilde{V}_m$.

As long as all collisions are good, and $t < \tau$, V and V' will coincide and $r_m(V', v) = r_m(V, v)$. Once $V' \neq V$ this is no longer true and the rates for (Ma) determined by step 1 will be given by $r_m(V, v) \neq r_m(V', v)$. Since V'_m is to be a realization of $\tilde{V}_m$, it should have the same rates as $\tilde{V}_m$, namely, $r_m(V', v)$. Thus step 1 needs some modification.

3. In order to obtain the correct rates we modify step 1 by ignoring some collisions [they produce no effect on (Ma)] and adding "extra collisions," depending on whether $r_m(V', v)$ is greater or less than $r_m(V, v)$:
   (a) The rate for the occurrence of these extra collisions is

$$R_m(V', V, v) = \begin{cases} \max(r_m(V', v) - r_m(V, v), 0) & \text{for} \quad t < \tau \\ r_m(V', v) & \text{for} \quad t \geqslant \tau \end{cases} \tag{6.2}$$

   (b) The probability that a collision of step 1 counts (i.e., is not ignored) is

$$p_m(V', V, v) = \begin{cases} \min\left(\dfrac{r_m(V', v)}{r_m(V, v)}, 1\right) & \text{for} \quad t < \tau \\ 0 & \text{for} \quad t \geqslant \tau \end{cases} \tag{6.3}$$

This modification, along with steps 1 and 2, yields a process V'_m governed by the rates $r_m(V', v)$ for $\tilde{V}_m$.

Note that the realization V'_m of $\tilde{V}_m$ has mechanical features in that it is tied to V_m by using most collisions of (Me) and it has a stochastic nature in that some of its collisions are purely governed by rates $r_m(V', v)$.

Let $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}_m)$ be a probability space on which V_m and V'_m satisfying the above description are realized: in particular $\bar{\mathbb{P}}_m(\{\bar{\omega} \in \bar{\Omega} \mid$

$V'_m(t, \bar{\omega}) \in A\}) = \bar{\mathbb{P}}_m(A)$ for all $A \in \mathscr{B}(D(I))$ and V'_m differs from V_m only by virtue of the rates $R_m(V', V, v)$, the probabilities $p_m(V', V, v)$, and bad collisions occurring for both (Me) and (Ma). $\bar{\Omega}$ can be viewed as a product space $\bar{\Omega} = \Omega \times H$, where the stochastic aspects are represented by H so that $V_m(\bar{\omega}) = V_m(\omega, h) = V_m(\omega)$ and $\bar{\mathbb{P}}_m(\cdot xh) = \mathbb{P}_m(\cdot)$.

We now have a good coupling:

Lemma 6.1. For all $\varepsilon > 0$

$$\lim_{m \to 0} \bar{\mathbb{P}}_m(\{\bar{\omega} \in \bar{\Omega} \mid \sup_{t \in I} |V'_{m,t}(\bar{\omega}) - V_{m,t}(\bar{\omega})| \geqslant \varepsilon\}) = 0 \tag{6.4}$$

Following Holley,[4] we prove the following lemma from which Lemma 6.1 follows by a "step-up" argument.

Lemma 6.2. If $t_0 \geqslant 0$ is such that

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{0 \leqslant t \leqslant t_0} |V'_{m,t}(\bar{\omega}) - V_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0 \tag{6.5}$$

for all $\varepsilon > 0$, then

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{0 \leqslant t \leqslant t_0 + z} |V'_{m,t}(\bar{\omega}) - V_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0 \tag{6.6}$$

for all $\varepsilon > 0$ where $z = 1/(84\rho V^0)$.

Remark 6.1. Lemma 6.1 follows easily from Lemma 6.2. Since $V'_m(0) = V_m(0)$, so that for $t_0 = 0$, the hypothesis of Lemma 6.2 is fulfilled, we obtain (6.4) for $I = [0, z]$. Apply Lemma 6.2 with $t_0 = z$. Then we obtain (6.4) for $I = [0, 2z]$. Iterating, we obtain (6.4) for any $I = [0, T]$.

We now turn to the proof of Lemma 6.2. Suppose t_0 satisfies (6.5). We may assume that $T > t_0 + z$. Given $\varepsilon > 0$, we introduce the stopping time

$$t_m^* = \inf\{t \geqslant 0 \mid |V'_m(t) - V_m(t)| \geqslant \varepsilon\} \tag{6.7}$$

Since $V'_m(t)$ and $V_m(t)$ are right continuous

$$|V'_m(t_m^*) - V_m(t_m^*)| \geqslant \varepsilon \tag{6.8}$$

and

$$|V'_m(s) - V_m(s)| < \varepsilon \qquad \text{for} \quad s < t_m^* \tag{6.9}$$

Let $G_m = \{\bar{\omega} \in \bar{\Omega} \mid \frac{9}{10}\bar{V}(t) < V'_m(t, \bar{\omega}) < \frac{11}{10}\bar{V}(t), \text{ for } 0 < t < T\}$. By Lemma 4.2, we have $\lim_{m \to 0} \bar{\mathbb{P}}_m(G_m) = 1$. What then must be shown is that

$$\lim_{m \to 0} \bar{\mathbb{P}}_m(\{t_m^* \leqslant t_0 + z; |V'_m(t_m^*) - V_m(t_m^*)| \geqslant \varepsilon\} \cap G_m) = 0 \tag{6.10}$$

From now on we assume $\varepsilon < \bar{V}(t)/10$. For $t < t_m^* \leqslant T$ on G_m we obtain $\frac{4}{5}\bar{V}(t) < V_m(t) < \frac{6}{5}\bar{V}(t)$.

Remark 6.2. If (Me) has a good collision, then on G_m for $t < t_m^* \leqslant T$ such a collision is always an initial collision. This may be seen by tracing the paths of (Me) and a colliding atom with velocity $v < \bar{V}(T)/4$ from the collision point backwards in time and noting that $V_m(t) > \frac{4}{5}\bar{V}(t) > \bar{V}(T)/4$. Moreover, for m sufficiently small, any such good collision produces a post-collision velocity larger than $\frac{6}{5}\bar{V}(t)$ for the atom, so that no future recollisions are possible. Thus on G_m for $t < t_m^* \leqslant T$, the only collisions which give rise to possible recollisions are the bad collisions where m is small.

We now compare the velocities of (Ma) and (Me). There are several effects which cause V'_m to differ from V_m:

1. The Markov process V'_m (mechanical process V_m) involves bad collisions with $v > V'$ $(v > V)$, in which the atoms overtake (Ma) $((Me))$ from behind. Let $E'_m(t)$ $(E_m(t))$ be the total change in $V'_m(V_m)$ on $[0, t]$ due to such collisions.

2. Let $S'_m(t)(S_m(t))$ be the total change in $V'_m(V_m)$ on $[0, t]$ due to bad collisions with $V' > v$ $(V > v)$, in which (Ma) $((Me))$ overtakes the atom from behind.

3. There is the change in V'_m directly due to "extra collisions" for (Ma) and the change in V_m due to good collisions which do not count in V'_m, the "extra collisions" for (Me). Let $A'(A)$ be the index set for the extra collisions for $(Ma)((Me))$ within $[t_0, t_m^*]$.

4. The effect on V' and the effect on V of those good collisions for (Me) which are also counted for V'_m depend, respectively, on the precollision velocities of (Ma) and (Me), which may differ. Let B be the index set for these collisions in $[t_0, t_m^*]$.

Thus, writing $W'(i)(W(i))$ for the change in the velocity of $(Ma)((Me))$ due to collision i, we have that for $t_m^* > t_0$

$$
\begin{aligned}
|V'_m(t_m^*) - V_m(t_m^*)| \\
&= \Big| E'_m(t_m^*) - E'_m(t_0) - (E_m(t_m^*) - E_m(t_0)) + S'_m(t_m^*) - S'_m(t_0) \\
&\quad - (S_m(t_m^*) - S_m(t_0)) + \sum_{i \in A'} W'_m(i) - \sum_{i \in A} W_m(i) \\
&\quad + \sum_{i \in B} (W'_m(i) - W_m(i)) + V'_m(t_0) - V_m(t_0) \Big| \\
&\leqslant \sum_{j=1}^{6} W_m^{(j)}(t_m^*) + |V'_m(t_0) - V_m(t_0)|
\end{aligned}
$$

where

$$W_m^{(1)}(t_m^*) = |E'_m(t_m^*) - E'_m(t_0)|$$

$$W_m^{(2)}(t_m^*) = |E_m(t_m^*) - E_m(t_0)|$$

$$W_m^{(3)}(t_m^*) = |S'_m(t_m^*) - S'_m(t_0)|$$

$$W_m^{(4)}(t_m^*) = |S_m(t_m^*) - S_m(t_0)|$$

$$W_m^{(5)}(t_m^*) = \sum_{i \in A'} |W'_m(i)| + \sum_{i \in A} |W_m(i)|$$

$$W_m^{(6)}(t_m^*) = \sum_{i \in B} |W'_m(i) - W_m(i)|$$

Then

$$\{|V'_m(t_m^*) - V_m(t_m^*)| \geqslant \varepsilon\} \subset \bigcup_{j=1}^{6} \left\{W_m^{(j)}(t_m^*) \geqslant \frac{\varepsilon}{7}\right\}$$
$$\cup \left\{|V'_m(t_0) - V_m(t_0)| \geqslant \frac{\varepsilon}{7}\right\} \cup \{t_m^* \leqslant t_0\}$$

By hypothesis, $\lim_{m \to 0} \bar{\mathbb{P}}_m(\{|V'_m(t_0) - V_m(t_0)| \geqslant \varepsilon/7\}) = 0$ and $\lim_{m \to 0} \bar{\mathbb{P}}_m\{t_m^* \leqslant t_0\} = 0$. So what needs to be shown is that

$$\lim_{m \to 0} \bar{\mathbb{P}}_m \left(G_m \cap \{t_m^* \leqslant t_0 + z\} \cap \left\{W_m^{(j)}(t_m^*) \geqslant \frac{\varepsilon}{7}\right\}\right) = 0 \tag{6.11}$$

for $j = 1, \dots, 6$.

The following estimates are for $\bar{\omega} \in G_m$, $t \leqslant t_m^*$, and $\varepsilon < \bar{V}(t)/10$.

$j = 1$. By the collision equation (2.5) the change in velocity of (Ma) due to a collision with an atom with velocity $v > V'$ satisfies

$$|\Delta V| \leqslant \frac{2m}{M} v$$

Thus

$$\bar{\mathbb{P}}_m \left(G_m \cap \{t_m^* < t_0 + z\} \cap \left\{W_m^{(1)}(t_m^*) \geqslant \frac{\varepsilon}{7}\right\}\right) < \bar{\mathbb{P}}_m \left(\sum_i \frac{2m}{M} v_i \geqslant \frac{\varepsilon}{7}\right)$$

where v_i is the velocity of the ith atom to hit (Ma) in time $[0, t_m^*]$ with $v > \frac{9}{10}\bar{V}(t)$.

By the properties of the Poisson random field and the Chebyshev inequality we find

$$\bar{\mathbb{P}}_m \left(\sum_i \frac{2m}{M} v_i \geqslant \frac{\varepsilon}{7}\right) \leqslant \frac{14m}{\varepsilon M} E\left(\sum_i v_i\right)$$

The rate for these collisions is bounded by $\rho v f_m(v)/m$. Therefore

$$\frac{14m}{\varepsilon M} E\left(\sum_i v_i\right) \leqslant \frac{14\rho}{\varepsilon M} \int_0^T \int_{(9/10)\bar{V}(T)}^{\infty} v^2 f_m(v)\, dv\, dt$$

$$= \frac{14\rho T}{\varepsilon M} \int_{(9/10)\bar{V}(T)}^{\infty} v^2 f_m(v)\, dv = O(m^{2\gamma})$$

$j=2$. Since (Me) can collide many times with the same atom, we here use conservation of momentum to estimate the effect of collisions from behind with an atom whose velocity prior to its first collision in $[0, t_m^*]$ is v. Let $v(t)$ be the velocity of this atom at time t. Let

$$|\Delta p| = m\, |v(t_m^*) - v(t_0)|$$

be the absolute total change in momentum of the atom during $[t_0, t_m^*]$. Since $v(t)$ is decreasing, it is easy to see that

$$-v \leqslant v(t_m^*) \leqslant v$$

so that

$$|\Delta p| \leqslant 2mv$$

and

$$|\Delta V| \leqslant \frac{2m}{M} v$$

Then, considering collisions for which $v \geqslant \frac{4}{5}\bar{V}(t)$, we conclude the estimate exactly as for $j=1$.

$j=3$. In this estimate (Ma) has collisions with atoms of velocities greater than $\bar{V}(t)/4$ but less than V'. In the prescription for V'_m, the rate for bad collisions for (Ma) is given by $r_m(V', v)$. On G_m an upper bound for this rate is given by $2\rho V^0 f_m(v)/\alpha$. We overestimate this effect by using this upper bound over the entire interval $[0, T]$, and integrating over all velocities greater than $\bar{V}(T)/4$. Let $N(T)$ be the number of collisions for (Ma) in which $2V^0 > V' > v > \bar{V}(T)/4$ in $[0, T]$. The change in the velocity of (Ma) as given in (2.5) is $\Delta V' = 2\alpha(V' - v) < 4\alpha V^0$. Therefore

$$\bar{\mathbb{P}}_m\left\{G_m \cap (t_m^* < t_0 + z) \cap \left[W_m^{(3)}(t_m^*) \geqslant \frac{\varepsilon}{7}\right]\right\}$$

$$\leqslant \bar{\mathbb{P}}_m\left\{\left[4\alpha V^0 N(T) \geqslant \frac{\varepsilon}{7}\right]\right\} \leqslant \frac{28\alpha V^0}{\varepsilon} E[N(T)]$$

$$\leqslant \frac{28\alpha V^0 T}{\varepsilon} \int_{\bar{V}(T)/4}^{\infty} \frac{\rho}{\alpha}(2V^0) f_m(v)\, dv = \frac{56\rho T (V^0)^2}{\varepsilon} \int_{\bar{V}(T)/4}^{\infty} f_m(v)\, dv$$

and with $\lambda = \bar{V}(T)/4$

$$\int_{v=\lambda}^{\infty} f_m(v)\,dv = \int_{u=m^{-\gamma}\lambda}^{\infty} f(u)\,du \xrightarrow[m \to 0]{} 0$$

$j=4$. Here we are concerned with the effect on the velocity of (Me) due to all initial collisions and possible recollisions with atoms of velocities greater than $\bar{V}(T)/4$ which are caught from behind by (Me).

Recall that on G_m such collisions can't take place at time t only if the atom has not yet acquired a velocity greater than $\frac{6}{5}\bar{V}(t)$. The absolute change in momentum during $[t_0, t_m^*]$ of a colliding atom whose velocity at time t is $v(t)$ is given by $|\Delta p| = m\,|v(t_m^*) - v(t_0)|$. Since $V \leqslant \frac{6}{5}V^0$ on G_m, we obtain from (2.5) that $v(t_m^*) \leqslant 3V^0$, so that $|\Delta p| \leqslant 3mV^0$. Thus, for the total change ΔV in the velocity of the molecule due to collisions with this atom we have $\Delta V \leqslant 3mM^{-1}V^0$. Thus

$$\bar{\mathbb{P}}_m \left\{ G_m \cap (t_m^* < t_0 + z) \cap \left[W_m^{(4)}(t_m^*) \geqslant \frac{\varepsilon}{7} \right] \right\} \leqslant \bar{\mathbb{P}}_m \left[3mM^{-1}V^0 N(T) \geqslant \frac{\varepsilon}{7} \right]$$

where $N(T)$ is the number of first collisions in $[0, T]$ between (Me) and atoms with velocities v with $\bar{V}(T)/4 < v < 2V^0$ for which $V < 2V^0$, the rate for which is bounded by $2\rho V^0 f_m(v)/\alpha$. The result therefore follows as for $j=3$.

$j=5$. For the rate of the occurrence of extra collisions including "extra collisions" for (Me) we have $(v < \bar{V}(T)/4)$

$$\tilde{R}_m(V', V, v) = |r_m(V', v) - r_m(V, v)|$$
$$= \frac{\rho}{\alpha}\,|V' - V|\, f_m(v)$$

Since $|V' - V| \leqslant \varepsilon$ for $t < t_m^*$ we consider the Poisson field X_m on the $t-v$ space determined by the rates

$$R_m^p(\varepsilon, v) = \frac{\rho}{\alpha}\,\varepsilon f_m(v)$$

which dominate the rate for extra collisions.

Let ΔV be the change in the velocity due to an (extra) collision either for (Ma) or (Me). From (2.5) we get that

$$|\Delta V| \leqslant 2\alpha(|v| + 2V^0)$$

Using the X_m process we obtain

$$\bar{\mathbb{P}}_m\left\{G^m \cap (t_m^* < t_0 + z) \cap \left[W_m^{(5)}(t_m^*) \geqslant \frac{\varepsilon}{7}\right]\right\}$$

$$\leqslant \text{Prob}\left[\sum_{i=1}^{N} 2\alpha(|v_i| + 2V^0) \geqslant \frac{\varepsilon}{7}\right] \tag{6.12}$$

where $|v_i|$ is the speed of the ith extra collision in $[t_0, t_0 + z)$ arising from the Poisson field X_m. Let E_m denote the expectation corresponding to X_m

$$E_m\left[\sum_{i=1}^{N} 2\alpha(|v_i| + 2V^0)\right] = 2\alpha E_m\left[\sum_{i=1}^{N} (|v_i| + 2V^0)\right]$$

$$= 2\alpha z\left[\int_{-\infty}^{\bar{V}(T)/4} \frac{\rho}{\alpha}\,\varepsilon(|v| + 2V^0)\, f_m(v)\, dv\right]$$

$$\leqslant 2z\varepsilon\rho(m^\gamma\langle |u|\rangle + 2V^0) \leqslant \frac{\varepsilon}{14}$$

for our choice of z and m sufficiently small.

Hence the r.h.s. of (6.12) can be estimated as follows. Let

$$J = \sum_{i=1}^{N} (|v_i| + 2V^0)$$

Then $\text{Prob}(2\alpha J \geqslant \varepsilon/7) \leqslant \text{Prob}[J \geqslant 2E_m(J)] = \text{Prob}[J - E_m(J) \geqslant E_m(J)] \leqslant E_m\{[J - E_m(J)]^2\}/[E_m(J)]^2$ by Chebyshev's inequality. Using correlation functions and some basic properties of Poisson random fields we have that for the X_m process

$$E_m(J) = (\langle |v|\rangle_m + 2V^0)\, E_m(N)$$

where N is the number of extra collisions in $[t_0, t_0 + z]$ arising from the X_m process, and

$$E_m(J^2) = E_m(N)\langle (|v| + 2V^0)^2\rangle_m + [E_m(N)]^2\langle |v| + 2V^0\rangle_m^2$$

Thus

$$E_m\{[J - E_m(J)]^2\}/[E_m(J)]^2 = \frac{1}{E_m(N)}\,\frac{\langle (m^\gamma\,|u| + 2V^0)^2\rangle}{\langle m^\gamma\,|u| + 2V^0\rangle^2} \xrightarrow[m\to 0]{} 0$$

since $E_m(N) \sim 1/m$ for the X_m process.

$j=6$. For the effect $W_i(W_i')$ of a good collision between atom i with velocity v_i and $(Me)[(Ma)]$ at time $s<t_m^*$ we have, using (2.5),

$$\begin{aligned}|W_i'-W_i| &= |V'(s^-)+2\alpha[v_i-V'(s^-)]-V'(s^-)\\ &\quad -\{V(s^-)+2\alpha[v_i-V(s^-)]-V(s^-)\}|\\ &=2\alpha\,|V'(s^-)-V(s^-)|\leqslant 2\alpha\varepsilon \qquad \text{by (6.9)}\end{aligned}$$

Hence $W_m^{(6)}(t_m^*)\leqslant 2\alpha\varepsilon N_m(z)$ where $N_m(z)$ is the number of collisions involving (Ma) in $[t_0, t_0+z]$. On G_m the total rate of collisions for the Markov process is bounded by $\rho(\langle|v|\rangle_m+2V^0)/\alpha$. Letting $\bar{N}_m(z)$ be the number of points in $[t_0, t_0+z]$ for the Poisson process with rate $\rho(\langle|v|\rangle_m+2V^0)/\alpha$, we obtain

$$\begin{aligned}&\bar{\mathbb{P}}_m\left\{G_m\cap(t_m^*<t_0+z)\cap\left[W_m^{(6)}(t_m^*)\geqslant\frac{\varepsilon}{7}\right]\right\}\\ &\quad\leqslant\bar{\mathbb{P}}_m\left\{\left[2\alpha\varepsilon N_m(z)\geqslant\frac{\varepsilon}{7}\right]\right\}=\bar{\mathbb{P}}_m\left\{\left[N_m(z)\geqslant\frac{1}{14\alpha}\right]\right\}\\ &\quad\leqslant\bar{\mathbb{P}}_m\left\{\left[\bar{N}_m(z)\geqslant\frac{1}{14\alpha}\right]\right\}\end{aligned}$$

Since

$$2E[\bar{N}_m(z)]=\frac{2z\rho}{\alpha}(\langle|v|\rangle_m+2V^0)\leqslant\frac{1}{14\alpha}$$

for our choice of z, and m sufficiently small, we have

$$\begin{aligned}\text{Prob}\left\{\left[\bar{N}_m(z)\geqslant\frac{1}{14\alpha}\right]\right\}&\leqslant\text{Prob}\{[\bar{N}_m(z)\geqslant 2E(\bar{N}_m(z))]\}\\ &=\text{Prob}\{[\bar{N}_m(z)-E(\bar{N}_m(z))\geqslant E(\bar{N}_m(z))]\}\\ &\leqslant E\{[\bar{N}_m(z)-E\bar{N}_m(z)]^2\}/[E\bar{N}_m(z)]^2\\ &=\frac{1}{E\bar{N}_m(z)}\xrightarrow[m\to 0]{}0\end{aligned}$$

This completes the proof of Lemma 6.2 and hence of Lemma 6.1 and Theorem 3.1.

We now turn to the proof of Theorem 3.2, which concerns the ξ process. Theorem 3.2 follows from Lemma 6.3 below which follows from Lemma 6.4.

We defined $\tilde{\xi}_m = (\tilde{V}_m - \bar{V}_m)/\sqrt{\alpha}$ and $\xi_m = (V_m - \bar{V}_m)/\sqrt{\alpha}$. We obtain a good coupling ξ'_m on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}}_m)$ satisfying

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{t \in I} |\xi'_{m,t}(\bar{\omega}) - \xi_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0$$

by using V'_m, namely, we let $\xi'_m = (V'_m - \bar{V}_m)/\sqrt{\alpha}$.

Lemma 6.3. For all $\varepsilon > 0$

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{t \in I} |\xi'_{m,t}(\bar{\omega}) - \xi_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0$$

Lemma 6.4. If $t_0 \geqslant 0$ is such that

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{0 \leqslant t \leqslant t_0} |\xi'_{m,t}(\bar{\omega}) - \xi_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0$$

for all $\varepsilon > 0$, then

$$\lim_{m \to 0} \bar{\mathbb{P}}_m\{[\bar{\omega} \in \bar{\Omega} \mid \sup_{0 \leqslant t \leqslant t_0 + z} |\xi'_{m,t}(\bar{\omega}) - \xi_{m,t}(\bar{\omega})| \geqslant \varepsilon]\} = 0$$

for all $\varepsilon > 0$, where $z = 1/(84\rho V^0)$.

That Lemma 6.4 implies Lemma 6.3 follows as in Remark 6.1. To prove Lemma 6.4 it will suffice to establish Lemma 6.2 modified by replacing ε by $\varepsilon_0 = \sqrt{\alpha}\,\varepsilon$, since $|\xi'_{m,t}(\bar{\omega}) - \xi_{m,t}(\bar{\omega})| \geqslant \varepsilon$ is equivalent to $|V'_{m,t}(\bar{\omega}) - V_{m,t}(\bar{\omega})| \geqslant \sqrt{\alpha}\,\varepsilon$. Now the final estimate for $j = 6$ did not involve ε, while the final estimate for $j = 5$ involves ε only through $E_m(N)$, which will now behave like $1/\sqrt{m}$. The final estimates of the remaining effects now assume the form

$$\frac{K}{\varepsilon_0} \int_{\bar{V}(T)/4}^{\infty} v^i f_m(v)\, dv$$

where $i = 0, 2$ and K is a constant, so we must show that these integrals are $o(\sqrt{m})$. To do this observe that for $\delta \in \mathbb{Z}^+$, $\lambda > 0$

$$\int_{v=\lambda}^{\infty} v^i f(v)\, dv \leqslant \int_{\lambda}^{\infty} \left(\frac{v}{\lambda}\right)^{\delta} v^i f(v)\, dv = \frac{1}{\lambda^{\delta}} \int_{\lambda}^{\infty} v^{\delta+i} f(v)\, dv \leqslant \frac{1}{\lambda^{\delta}} \int_{0}^{\infty} v^{\delta+i} f(v)\, dv$$

Thus

$$\frac{K}{\varepsilon_0}\int_{\bar{V}(T)/4}^{\infty} v^i f_m(v)\,dv = \frac{K}{\sqrt{\alpha}\,\varepsilon}\int_{m^{-\gamma}\bar{V}(T)/4}^{\infty} (m^{\gamma}u)^i f(u)\,du$$
$$\leqslant \frac{K}{\sqrt{\alpha}\,\varepsilon}\left(\frac{4m^{\gamma}}{\bar{V}(T)}\right)^{\delta} m^{\gamma i}\int_0^{\infty} u^{\delta+i} f(u)\,du$$
$$= o(\sqrt{m})$$

for $\gamma\delta > \frac{1}{2}$, since all moments of $f(v)$ are finite. This completes the proof.

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REFERENCES

1. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
2. D. Dürr, S. Goldstein, and J. L. Lebowitz, *Comm. Math. Phys.* **78**:507–530 (1981).
3. E. B. Dynkin, *Markov Processes I, II* (Springer, Berlin, 1965).
4. R. Holley, *Z. Wahrsch. Verw. Gebiete* **17**:181–219 (1971).
5. T. Kurz, *Ann. Probability* **4**:618–642 (1975).
6. B. Miller and W. Stein, *J. Stat. Phys.* **26**(3):539–553 (1981).
7. N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, (North-Holland, Amsterdam, 1981).
8. L. N. Vaserstein, *Comm. Math. Phys.* **69**:31–56 (1979).